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Solutions of the Generalized Weierstrass Representation in Four-Dimensional Euclidean Space

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Abstract

Several classes of solutions of the generalized Weierstrass system, which induces constant mean curvature surfaces into four-dimensional Euclidean space are constructed. A gauge transformation allows us to simplify the system considered and derive factorized classes of solutions. A reduction of the generalized Weierstrass system to decoupled CP^1 sigma models is also considered. A new procedure for constructing certain classes of solutions, including elementary solutions (kinks and bumps) and multisoliton solutions is described in detail. The constant mean curvature surfaces associated with different types of solutions are presented. Some physical interpretations of the results obtained in the area of string theory are given.

1 Introduction

It has been shown [1] that Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [2] and Enneper [3] in the middle of the nineteenth century on systems inducing minimal surfaces in $\mathbb{R}^3$. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory [4], statistical physics [5], chemical physics, fluid dynamics and membranes [6], minimal surfaces play an essential role. More recently it is worth mentioning that works by Kenmotsu [7], Hoffmann [8], Osserman [9], Budinich [10], Konopelchenko [11, 12] and Bobenko [13, 14] have

made very significant contributions to constructing minimal surfaces in a systematic way and to understanding their intrinsic geometric properties as well as their integrable dynamics. The type of extension of the Weierstrass representation which has been useful in three-dimensional applications to multidimensional spaces will continue to generate many additional applications to physics and mathematics. According to [15] integrable deformations of surfaces are generated by the Davey–Stewartson hierarchy of $2+1$ dimensional soliton equations. These deformations of surfaces inherit all the remarkable properties of soliton equations. Geometrically such deformations are characterised by the invariance of an infinite set of functionals over surfaces, the simplest being the Willmore functional.

In recent years, major developments in the area of the low dimensional sigma model have been shown [16] to be of use in generating two-dimensional surfaces immersed in multidimensional-space. There are links between this model and other models, such as the non Abelian Chern–Simons theories which have been of interest recently in condensed matter physics [17]. In fact the Chern–Simons gauged Landau–Ginsburg model plays the essential role of an effective theory for the Fractional Quantum Hall Effect. There exists a link between non Abelian Chern–Simons theories and the nonlinear sigma model, which is related to minimal surfaces. For example, in [18], a simple method was proposed to obtain completely integrable systems in $(2+1)$-dimensions from classes of non Abelian Chern–Simons field theory. In this sense completely integrable systems are seen as particular gauge choices in which the theory is formulated. Moreover linear spectral problems are naturally related to the geometrical constraints imposed on the target space. Among several possibilities for building up integrable deformations of $(2+1)$-dimensional surfaces, multidimensional integrable spin field systems can be used to realize integrable deformations of surfaces. A more general $(2+1)$-dimensional integrable spin model is described by the pair of equations [17],

$$\vec{S}_t + \vec{S} \wedge \left\{ (b+1)\vec{S}_{ss} - b\vec{S} \right\} + bu_t\vec{S}_t + (b+1)u_s\vec{S}_s = 0,$$
$$u_{st} = \vec{S} \cdot (\vec{S}_s \wedge \vec{S}_t),$$

where s and t are real or complex variables, b a real constant, $\vec{S} = (S_1, S_2, S_3)$ is the spin field vector, $\vec{S}^2 = 1$ and u is a scalar function. These represent one of the $(2+1)$-dimensional integrable generalizations of the isotropic Landau–Lifshitz equation

$$\vec{S}_t = \vec{S} \wedge \vec{S}_{xx}.$$

The process of unifying gravity, supersymmetry and gauge theories leads to supergravity theories such that the number of supersymmetries goes from one to eight. Unfortunately quantum Einstein gravity is non renormalizable [4, 5]. A quantum theory of gravity should therefore be a nonlocal quantum field theory. Modern superstring theory draws together many concepts of field theory, for example gauge symmetry, supersymmetry, effective actions and the nonlinear sigma models. An extension of the Polyakov string integral over multidimensional spaces would be of great interest [15]. In fact the string action in a non trivial gravitational background takes the form of a non linear sigma model or a generalization of it. Conformal invariance plays a fundamental role in perturbative string theory and results in deep connections between strings and the nonlinear sigma model. Of particular physical interest is the geometrical nature of the interaction in the nonlinear

sigma model which has consequences such as the geometrical nature of the counterterms which are required for renormalizability, the existence of topologically non trivial field configurations such as solitons and gauge invariance of four-dimensional quantum field theories.

In this paper we construct several classes of solutions of the Weierstrass representation inducing constant mean curvature (CMC) surfaces immersed in Euclidean four-dimensional space. This paper is an extension of previous papers [19, 20] which concern the Weierstrass representation for CMC-surfaces immersed in Euclidean three-dimensional space. This representation has recently been introduced by Konopelchenko and Landolfi [1] and will be referred to as the KL system. Their formulas are the starting point of our analysis, namely, they consider a first order nonlinear system of two-dimensional Dirac type equations for four complex valued functions ψ_α and φ_α ($\alpha = 1, 2$). This system can be written as follows

$$\begin{aligned} &\partial\psi_\alpha = p\varphi_\alpha, \qquad \bar\partial\varphi_\alpha = -p\psi_\alpha, \qquad \alpha = 1, 2, \\ &p = \sqrt{u_1 u_2}, \qquad u_\alpha = |\psi_\alpha|^2 + |\varphi_\alpha|^2 \end{aligned} \tag{1.1}$$

and their complex conjugate equations. We denote the derivatives in abbreviated form by $\partial = \partial/\partial z$, $\bar\partial = \partial/\partial\bar z$ and the bar denotes complex conjugation. Note that eight of sixteen first derivatives of the fields ψ_α and φ_α appearing in the KL system (1.1) are given in terms of the complex functions ψ_α and φ_α only. These functions ψ_α and φ_α are invariant under the multiplication factor minus one. The system (1.1) possesses several conservation laws such as

$$\partial(\psi_\alpha\psi_\beta) + \bar\partial(\varphi_\alpha\varphi_\beta) = 0, \qquad \partial(\psi_\alpha\bar\varphi_\beta) - \bar\partial(\varphi_\alpha\bar\psi_\beta) = 0, \qquad \alpha \neq \beta = 1, 2. \tag{1.2}$$

As a consequence of these conserved quantities there exist four real valued functions $X_i(z, \bar z)$, $i = 1, \ldots, 4$, which can be interpreted as coordinates for a surface immersed in Euclidean space $\mathbb{R}^4$. The coordinates of the position vector $\mathbf{X} = \left(X^1, X^2, X^3, X^4\right)$ of a CMC-surface in $\mathbb{R}^4$ are given by [1]

$$\begin{aligned} X^1 &= \frac{i}{2}\int_\gamma \left[(\bar\psi_1\bar\psi_2 + \varphi_1\varphi_2)\,dz' - (\psi_1\psi_2 + \bar\varphi_1\bar\varphi_2)\,d\bar z'\right], \\ X^2 &= \frac{1}{2}\int_\gamma \left[(\bar\psi_1\bar\psi_2 - \varphi_1\varphi_2)\,dz' + (\psi_1\psi_2 - \bar\varphi_1\bar\varphi_2)\,d\bar z'\right], \\ X^3 &= -\frac{1}{2}\int_\gamma \left[(\bar\psi_1\varphi_2 + \bar\psi_2\varphi_1)\,dz' + (\psi_1\bar\varphi_2 + \psi_2\bar\varphi_1)\,d\bar z'\right], \\ X^4 &= \frac{i}{2}\int_\gamma \left[(\bar\psi_1\varphi_2 - \bar\psi_2\varphi_1)\,dz' - (\psi_1\bar\varphi_2 - \psi_2\bar\varphi_1)\,d\bar z'\right], \end{aligned} \tag{1.3}$$

where γ is any closed contour in the complex plane $\mathbb{C}$. Due to the conservation laws (1.2) the integrals appearing in system (1.3) do not depend upon the trajectory of contour γ in $\mathbb{C}$, but depend upon the endpoints. The differentials of equations (1.3) are exact ones. The mean and the Gaussian curvatures and the first and second fundamental forms of the surface immersed in $\mathbb{R}^4$ are given by [1]

$$\begin{aligned} &\mathbf{H}^2 = 4\frac{|p|^2}{u_1 u_2}, \qquad K = -p^{-2}\partial\bar\partial\ln p, \\ &ds^2 = u_1 u_2\,dz\,d\bar z, \qquad II = (\partial^2\mathbf{r}|\mathbf{n})dz^2 + (\partial\bar\partial\mathbf{r}|\bar{\mathbf{n}})dz\,d\bar z + (\bar\partial^2\mathbf{r}|\mathbf{n})d\bar z^2 \end{aligned} \tag{1.4}$$

in conformal coordinates, respectively. Here the vector $\mathbf{n}$ is the unit normal vector to a surface which satisfies $(\partial \mathbf{r}|\mathbf{n}) = 0$, $(\bar{\partial}\mathbf{r}|\mathbf{n}) = 0$, $\mathbf{n}^2 = 1$ and the bracket $(\ |\ )$ denotes the standard scalar product in $\mathbb{R}^4$.

In our investigations it is more convenient to introduce two new dependent complex variables which link the Weierstrass representation with a second order overdetermined system of PDEs. This link allows us to establish several useful transformations in order to simplify the structure of the KL system (1.1) and to construct several classes of solutions.

We define two new complex variables given by

$$\xi_\alpha = \frac{\psi_\alpha}{\bar{\varphi}_\alpha}, \qquad \alpha = 1, 2. \tag{1.5}$$

The equations (1.3) can be written in equivalent form [1] in terms of ξ_α as

$$\mathbf{X} = \int^z \mathrm{Re}\,(\vartheta \mathbf{G}\, dz), \tag{1.6}$$

where the functions ϑ and $G(z, \bar{z})$ are given by

$$\begin{aligned} &\vartheta^2 = -\frac{4\partial\xi_1\partial\xi_2}{\mathbf{H}^2(1+|\xi_1|^2)^2(1+|\xi_2|^2)^2}, \\ &\mathbf{G}(z,\bar{z}) = [(1+\bar{\xi}_1\bar{\xi}_2), i(1-\bar{\xi}_1\bar{\xi}_2), i(\bar{\xi}_1+\bar{\xi}_2), \bar{\xi}_1-\bar{\xi}_2] \end{aligned} \tag{1.7}$$

and the complex functions ξ_1 and ξ_2 obey the second order partial differential equation

$$-2\partial(\ln|\mathbf{H}|) + \frac{\partial\bar{\partial}\bar{\xi}_1}{\bar{\partial}\bar{\xi}_1} - \frac{2\xi_1\partial\bar{\xi}_1}{1+|\xi_1|^2} + \frac{\partial\bar{\partial}\bar{\xi}_2}{\bar{\partial}\bar{\xi}_2} - \frac{2\xi_2\partial\bar{\xi}_2}{1+|\xi_2|^2} = 0. \tag{1.8}$$

In particular, if $\psi_2 = \epsilon\psi_1$ and $\varphi_2 = \epsilon\varphi_1$ for $\epsilon = \pm 1$ hold, then the KL system (1.1) is reduced to the generalized Weierstrass (GW) system inducing CMC-surfaces in $\mathbb{R}^3$

$$\partial\psi_1 = p\varphi_1, \qquad \bar{\partial}\varphi_1 = -p\psi_1, \qquad p = |\psi_1|^2 + |\varphi_1|^2. \tag{1.9}$$

Equation (1.8) then becomes the equation for the CP^1 sigma model

$$\partial\bar{\partial}\xi - \frac{2\bar{\xi}}{1+|\xi|^2}\partial\xi\bar{\partial}\xi = 0 \tag{1.10}$$

and its conjugate, since $\xi_1 = \xi_2 = \xi$ by (1.5). These limits characterize the properties of solutions of the KL system (1.1).

In this paper we examine certain algebraic and differential constraints of the first order, compatible with the initial system of PDEs (1.1), which allow us to simplify its structure. In particular we focus upon constructing several classes of multisoliton solutions of system (1.1) which have not been found up to now. In some cases new, interesting, CMC surfaces are found in explicit form.

The paper is organized as follows. In Section 2 we perform a reduction of the original system (1.1) to a certain overdetermined system of PDEs. This overdetermined system allows us to simplify the structure of the KL system by performing a rational transformation for the functions ψ_α and φ_α in (1.1). In Section 3 we introduce a gauge transformation for the KL system and we discuss the possibility of factorization of the associated KL system.

Furthermore a new procedure for constructing solutions to the KL system is proposed. We formulate several useful propositions for building certain classes of solutions of the KL system. Section 4 deals with a reduction of the KL system to decoupled CP^1 sigma models. In Section 5 we apply these propositions in order to construct several explicit solutions and their superpositions. Based on these propositions we are able to generate new classes of multisoliton solutions and find their associated CMC surfaces. Section 6 contains a simple example related to classical string configurations in $\mathbb{R}^4$ and possible future developments.

2 A system associated with the Konopelchenko–Landolfi system

Now we introduce a new system associated with (1.1) which allows the construction of several classes of solutions for the KL system including elementary and multisoliton solutions, which are presented in Section 5.

Proposition 1. *If the complex valued functions ψ_α and φ_α are solutions of KL system (1.1), then the rational functions defined by (1.5) are solutions of the following overdetermined system,*

$$\frac{(\partial\xi_1)(\bar\partial\bar\xi_1)}{(1+|\xi_1|^2)^2} - \frac{(\partial\xi_2)(\bar\partial\bar\xi_2)}{(1+|\xi_2|^2)^2} = 0, \tag{2.1}$$

$$\begin{aligned} &-\frac{\partial\bar\partial\partial\xi_\alpha}{2\partial\xi_\alpha} + \frac{\bar\partial\partial\xi_\alpha}{2(\partial\xi_\alpha)^2}\partial^2\xi_\alpha - \frac{\partial\bar\partial\bar\xi_\alpha}{1+|\xi_\alpha|^2}\xi_\alpha + \frac{\xi_\alpha^2(\partial\bar\xi_\alpha)(\bar\partial\bar\xi_\alpha)}{(1+|\xi_\alpha|^2)^2} + \frac{\bar\partial\bar\partial\partial\xi_\alpha}{2\bar\partial\bar\xi_\alpha} \\ &\qquad - \frac{\bar\partial\partial\bar\xi_\alpha}{2(\bar\partial\bar\xi_\alpha)^2}\bar\partial^2\bar\xi_\alpha + \frac{\bar\partial\partial\xi_\alpha}{1+|\xi_\alpha|^2}\bar\xi_\alpha - \frac{\bar\xi_\alpha^2(\bar\partial\xi_\alpha)(\partial\xi_\alpha)}{(1+|\xi_\alpha|^2)^2} = 0, \qquad \alpha = 1,2. \end{aligned} \tag{2.2}$$

Proof. In fact, from (1.1) and making use of transformation (1.5), we get

$$u_\alpha = |\psi_\alpha|^2 + |\varphi_\alpha|^2 = |\varphi_\alpha|^2\left(1+|\xi_\alpha|^2\right), \qquad \alpha = 1,2. \tag{2.3}$$

By differentiation of equations (1.5) with respect to ∂ and using system (1.1), we obtain

$$\partial\xi_\alpha = p\,\frac{u_\alpha}{\bar\varphi_\alpha^2} = p\frac{\varphi_\alpha}{\bar\varphi_\alpha}\left(1+|\xi_\alpha|^2\right). \tag{2.4}$$

Taking the ratio of (2.4) with its complex conjugate, we get

$$\frac{\partial\xi_\alpha}{\bar\partial\bar\xi_\alpha} = \frac{\varphi_\alpha^2}{\bar\varphi_\alpha^2}. \tag{2.5}$$

Multiplying equation (2.4) by its complex conjugate and solving for p^2, we can express p^2 in terms of ξ_α as

$$(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha) = p^2\left(1+|\xi_\alpha|^2\right)^2. \tag{2.6}$$

So we have

$$p^2 = \frac{(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha)}{\left(1+|\xi_\alpha|^2\right)^2}, \qquad \alpha = 1,2. \tag{2.7}$$

Equating equations (2.4) for $\alpha = 1, 2$ we obtain expression (2.1). Using equation (2.5) we can write equations (2.7) in equivalent form

$$(i) \quad p^2 = \frac{\varphi_\alpha^2 (\bar{\partial}\bar{\xi}_\alpha)^2}{\bar{\varphi}_\alpha^2 (1 + |\xi_\alpha|^2)^2}, \qquad (ii) \quad p^2 = \frac{\bar{\varphi}_\alpha^2 (\partial \xi_\alpha)^2}{\varphi_\alpha^2 (1 + |\xi_\alpha|^2)^2}, \qquad \alpha = 1, 2. \tag{2.8}$$

Elimination of p^2 from (2.8), for different values of $\alpha = 1, 2$, leads to equation (2.1). Eliminating p^2 from equations (2.7) with $\alpha = 1$ and (2.8ii) with $\alpha = 2$ we get the following relation

$$\varphi_2^2 \frac{(\partial \xi_1)(\bar{\partial}\bar{\xi}_1)}{(1 + |\xi_1|^2)^2} - \bar{\varphi}_2^2 \frac{(\partial \xi_2)^2}{(1 + |\xi_2|^2)^2} = 0. \tag{2.9}$$

Similarly, from equations (2.7) with $\alpha = 2$ and equations (2.8ii) with $\alpha = 2$, we have

$$\varphi_2^2 (\bar{\partial}\bar{\xi}_2) - \bar{\varphi}_2^2 (\partial \xi_2) = 0. \tag{2.10}$$

Equations (2.9) and (2.10) have a nontrivial solution for functions φ_2^2 and $\bar{\varphi}_2^2$ if the determinant of their coefficients vanishes. This condition turns out to be exactly the condition (2.1). Elimination of p^2 from equations (2.7) and (2.8i) with $\alpha = 1$ and, next, elimination of p^2 from (2.7) with $\alpha = 2$ and (2.8ii) with $\alpha = 1$ leads to

$$\varphi_1^2 (\bar{\partial}\bar{\xi}_1) - \bar{\varphi}_1^2 (\partial \xi_1) = 0, \tag{2.11}$$

$$\varphi_1^2 \frac{(\partial \xi_2)(\bar{\partial}\bar{\xi}_2)}{(1 + |\xi_2|^2)^2} - \bar{\varphi}_1^2 \frac{(\partial \xi_1)^2}{(1 + |\xi_1|^2)^2} = 0. \tag{2.12}$$

The condition for the existence of nontrivial solutions for φ_1^2 and $\bar{\varphi}_1^2$ of equations (2.11) and (2.12) is reduced to condition (2.1). This means that we can take into account only two equations, say (2.10) and (2.11), from the systems of equations (2.9)–(2.12) since the determinant of their coefficients vanishes whenever (2.1) holds. This implies that equations (2.9)–(2.12) are linearly dependent. A general solution for the system (2.10) and (2.11) has the form

$$\varphi_\alpha = a_\alpha (\partial \xi_\alpha)^{1/2}, \qquad \bar{\varphi}_\alpha = \bar{a}_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2}. \tag{2.13}$$

For (2.13) to be consistent with (2.9)–(2.12) the functions $a_\alpha = a_\alpha(z, \bar{z})$ are real-valued so that $a_\alpha = \bar{a}_\alpha$. Substituting (2.13) into (1.5) we obtain

$$\psi_\alpha = a_\alpha \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2}, \qquad \bar{\psi}_\alpha = a_\alpha \bar{\xi}_\alpha (\partial \xi_\alpha)^{1/2}. \tag{2.14}$$

Differentiating (2.13) with respect to $\bar{\partial}$ and substituting into the KL system (1.1), we obtain

$$\begin{aligned} &(\bar{\partial} a_\alpha)(\partial \xi_\alpha)^{1/2} + \frac{1}{2} a_\alpha (\partial \xi_\alpha)^{-1/2} \partial \bar{\partial} \xi_\alpha \\ &\qquad = -p \xi_\alpha a_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2} = -\frac{(\partial \xi_\alpha)^{1/2} (\bar{\partial}\bar{\xi}_\alpha)^{1/2}}{1 + |\xi_\alpha|^2} \xi_\alpha a_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2}. \end{aligned}$$

This means that the functions ξ_α satisfy the second order differential equation

$$\partial \bar{\partial} \xi_\alpha + 2(\bar{\partial} \ln a_\alpha)(\partial \xi_\alpha) = -2 \frac{(\partial \xi_\alpha)(\bar{\partial}\bar{\xi}_\alpha)}{1 + |\xi_\alpha|^2} \xi_\alpha. \tag{2.15}$$

Similarly differentiation of (2.14) with respect to ∂ and use of the KL system (1.1) yields the complex conjugate of (2.15), namely,

$$\partial\bar{\partial}\bar{\xi}_\alpha + 2(\partial\,\ln a_\alpha)(\bar{\partial}\bar{\xi}_\alpha) = -2\frac{(\partial\xi_\alpha)(\bar{\partial}\bar{\xi}_\alpha)}{1+|\xi_\alpha|^2}\bar{\xi}_\alpha.$$

Equations (2.15) and their conjugates can be solved for the quantities $\bar{\partial}(\ln a_\alpha)$ and $\partial(\ln a_\alpha)$, respectively,

$$\begin{aligned} &(i) \quad \bar{\partial}\ln a_\alpha = -\frac{\bar{\partial}\partial\xi_\alpha}{2\partial\xi_\alpha} - \frac{\bar{\partial}\bar{\xi}_\alpha}{1+|\xi_\alpha|^2}\xi_\alpha, \\ &(ii) \quad \partial\ln a_\alpha = -\frac{\bar{\partial}\partial\bar{\xi}_\alpha}{2\bar{\partial}\bar{\xi}_\alpha} - \frac{\partial\xi_\alpha}{1+|\xi_\alpha|^2}\bar{\xi}_\alpha. \end{aligned} \tag{2.16}$$

By differentiation of (2.16i) with respect to ∂ and (2.16ii) with respect to $\bar{\partial}$, the compatibility condition for these derivatives results in (2.2). Thus, for a given solution ξ_α of system (2.1) and (2.2), the function a_α is determined uniquely by equations (2.16). By use of (2.15) in (1.8) the second derivatives can be eliminated from (1.8) to obtain an additional constraint on the functions a_α,

$$\partial\ln a_1 + \frac{1}{1+|\xi_1|^2}(\bar{\xi}_1\partial\xi_1 + \xi_1\partial\bar{\xi}_1) + \partial\ln a_2 + \frac{1}{1+|\xi_2|^2}(\bar{\xi}_2\partial\xi_2 + \xi_2\partial\bar{\xi}_2) = 0 \tag{2.17}$$

and its complex conjugate. By virtue of (1.8) the condition (2.17) is a necessary condition for the mean curvature H to be constant, according to (1.4). ■

Note that, if the real-valued functions a_α are holomorphic functions, then a_α are constant and (2.15) reduces to two decoupled CP^1 sigma model equations (1.10). The converse Proposition is also true.

Proposition 2. *Suppose that the complex valued functions ξ_α are solutions of the overdetermined system (2.1) and (2.2) and that the real-valued functions a_α are solutions of equations (2.16) and (2.17). Then the complex functions φ_α and ψ_α determined by*

$$\varphi_\alpha = a_\alpha(\partial\xi_\alpha)^{1/2}, \qquad \psi_\alpha = a_\alpha\xi_\alpha(\bar{\partial}\bar{\xi}_\alpha)^{1/2}, \qquad \alpha = 1,2, \tag{2.18}$$

are solutions of the KL system (1.1).

Proof. For a given solution of (2.1), (2.2) we assume that the functions a_α are consistent with the compatibility conditions (2.16) and (2.17). Differentiating φ_α in (2.18) with respect to $\bar{\partial}$ we obtain

$$\bar{\partial}\varphi_\alpha = (\bar{\partial}a_\alpha)(\partial\xi_\alpha)^{1/2} + \frac{1}{2}a_\alpha(\partial\xi_\alpha)^{-1/2}\bar{\partial}\partial\xi_\alpha. \tag{2.19}$$

Using (2.16i) we can eliminate the first derivative of a_α in equation (2.19) to obtain

$$\bar{\partial}\varphi_\alpha = \left(-\frac{\bar{\partial}\partial\xi_\alpha}{2(\partial\xi_\alpha)}a_\alpha - \frac{\bar{\partial}\bar{\xi}_\alpha}{1+|\xi_\alpha|^2}\xi_\alpha a_\alpha\right)(\partial\xi_\alpha)^{1/2} + \frac{1}{2}a_\alpha\,\frac{\bar{\partial}\partial\xi_\alpha}{(\partial\xi_\alpha)^{1/2}}.$$

Next, making use of (2.7) and (2.18), we get

$$\bar{\partial}\varphi_\alpha = -\frac{(\bar{\partial}\bar{\xi}_\alpha)^{1/2}(\partial\xi_\alpha)^{1/2}}{1+|\xi_\alpha|^2} a_\alpha \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2} = -p\psi_\alpha.$$

So the functions φ_α satisfy the first equation in (1.1). Differentiating ψ_α with respect to ∂ in (2.18), we obtain

$$\bar{\partial}\psi_\alpha = (\partial\xi_\alpha)a_\alpha(\bar{\partial}\bar{\xi}_\alpha)^{1/2} + \xi_\alpha(\partial a_\alpha)(\bar{\partial}\bar{\xi}_\alpha)^{1/2} + \frac{1}{2}\xi_\alpha a_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{-1/2}\partial\bar{\partial}\bar{\xi}_\alpha. \tag{2.20}$$

Similarly, using equations (2.16ii) and (2.7), we can eliminate the first derivatives in equation (2.20) and (2.18). We have

$$\begin{aligned}\bar{\partial}\psi_\alpha &= (\partial\xi_\alpha)a_\alpha(\bar{\partial}\bar{\xi}_\alpha)^{1/2} + \xi_\alpha(\bar{\partial}\bar{\xi}_\alpha)^{1/2}\left(-\frac{\bar{\partial}\partial\bar{\xi}_\alpha}{2(\bar{\partial}\bar{\xi}_\alpha)} - \frac{\partial\xi_\alpha}{1+|\xi_\alpha|^2}\bar{\xi}_\alpha\right)a_\alpha \\ &\quad + \frac{1}{2}\xi_\alpha a_\alpha(\bar{\partial}\bar{\xi}_\alpha)^{-1/2}\partial\bar{\partial}\bar{\xi}_\alpha = \frac{(\partial\xi_\alpha)^{1/2}(\bar{\partial}\bar{\xi}_\alpha)^{1/2}}{1+|\xi_\alpha|^2}a_\alpha(\partial\xi_\alpha)^{1/2} = p\varphi_\alpha.\end{aligned}$$

This completes the proof. ■

3 Gauge transformations and factorization of the associated Konopelchenko–Landolfi system

Now, we discuss certain new classes of multisoliton solutions of the KL system (1.1) which can be obtained directly by applying the transformation (1.5). Firstly, we demonstrate that the KL system (1.1) admits a gauge transformation. We introduce a new set of complex functions, $\kappa_\alpha, \tau_\alpha : \mathbb{C} \to \mathbb{C}$, which are related to the complex functions ψ_α and φ_α by

$$\psi_\alpha = f_\alpha(z,\bar{z})\kappa_\alpha, \qquad \bar{\varphi}_\alpha = f_\alpha(z,\bar{z})\bar{\tau}_\alpha \tag{3.1}$$

for any complex functions $f_\alpha : \mathbb{C} \to \mathbb{C}$.

From equations (1.5), since ψ_α, $\bar{\varphi}_\alpha$ appear as a ratio, it is evident that the transformation (3.1) leaves the functions ξ_α invariant

$$\xi_\alpha = \frac{\kappa_\alpha}{\bar{\tau}_\alpha}. \tag{3.2}$$

This means that there exists a freedom which resembles a type of gauge freedom in the definition of the functions, ξ_α, since the numerator and denominator of (1.5) can be multiplied by any complex function. The main point is that it is not required that the set of functions κ_α and $\bar{\tau}_\alpha$ satisfy the original system (1.1), but that the ratio of κ_α over $\bar{\tau}_\alpha$ has to satisfy the system (2.1) and (2.2).

Proposition 3. *Suppose that for any real holomorphic functions, g_α, the complex functions, κ_α, and, τ_α, are related to the complex functions, ψ_α, and, φ_α, as follows*

$$\kappa_\alpha = g_\alpha\psi_\alpha, \qquad \tau_\alpha = g_\alpha\varphi_\alpha, \qquad \partial g_\alpha = 0. \tag{3.3}$$

Then the functions κ_α and τ_α are solutions of the KL system (1.1) and have the form given by (2.14) provided that the functions ξ_α are solutions of (2.1) and (2.2) and the real valued functions $a_\alpha(z,\bar{z})$ have to satisfy the conditions (2.16) and (2.17).

Proof. Note, from the fact that ξ_α in (3.2) is invariant under the gauge function f_α, it follows that the function p given by (2.7) is invariant as well.

From (3.3) and using (2.14) we can write that

$$\kappa_\alpha = g_\alpha a_\alpha \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2}, \qquad \tau_\alpha = g_\alpha a_\alpha (\partial \xi_\alpha)^{1/2}. \tag{3.4}$$

Differentiating κ_α with respect to ∂, we obtain

$$\begin{aligned}\partial\kappa_\alpha &= (\partial g_\alpha) a_\alpha \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2} + g_\alpha (\partial a_\alpha) \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2} \\ &\quad + g_\alpha a_\alpha (\partial\xi_\alpha)(\bar{\partial}\bar{\xi}_\alpha)^{1/2} + \frac{1}{2} g_\alpha a_\alpha \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{-1/2} \partial\bar{\partial}\bar{\xi}_\alpha\end{aligned} \tag{3.5}$$

Substitution of the derivative ∂a_α obtained from (2.16) into (3.5), leads to (3.5) becoming

$$\begin{aligned}\partial\kappa_\alpha &= (\partial g_\alpha) a_\alpha \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2} - g_\alpha a_\alpha \frac{(\partial\xi_\alpha)(\bar{\partial}\bar{\xi}_\alpha)^{1/2}}{1+|\xi_\alpha|^2} |\xi_\alpha|^2 + g_\alpha a_\alpha (\partial\xi_\alpha)(\bar{\partial}\bar{\xi}_\alpha)^{1/2} \\ &= (\partial g_\alpha) a_\alpha \xi_\alpha (\bar{\partial}\bar{\xi}_\alpha)^{1/2} + \left(\frac{(\partial\xi_\alpha)^{1/2}(\bar{\partial}\bar{\xi}_\alpha)^{1/2}}{1+|\xi_\alpha|^2} \right) \left(g_\alpha a_\alpha (\partial\xi_\alpha)^{1/2} \right).\end{aligned} \tag{3.6}$$

Clearly, if $\partial g_\alpha = 0$, equation (3.6) takes the form of the first equation of the KL system (1.1), $\partial\kappa_\alpha = p\tau_\alpha$, as required.

Differentiating τ_α with respect to $\bar{\partial}$, we obtain

$$\begin{aligned}\bar{\partial}\tau_\alpha &= \bar{\partial} g_\alpha\, a_\alpha (\partial\xi_\alpha)^{1/2} + g_\alpha (\bar{\partial} a_\alpha)(\partial\xi_\alpha)^{1/2} + \frac{1}{2}(\partial\xi_\alpha)^{-1/2}(\bar{\partial}\partial\xi_\alpha) g_\alpha a_\alpha \\ &= a_\alpha (\bar{\partial} g_\alpha)(\partial\xi_\alpha)^{1/2} - g_\alpha a_\alpha \frac{(\partial\xi_\alpha)^{1/2}\bar{\partial}\bar{\xi}_\alpha}{1+|\xi_\alpha|^2} \xi_\alpha.\end{aligned}$$

Since g_α are holomorphic functions, the above equations simplify to

$$\bar{\partial}\tau_\alpha = -\frac{(\partial\xi_\alpha)^{1/2}(\bar{\partial}\bar{\xi}_\alpha)^{1/2}}{1+|\xi_\alpha|^2} \left(g_\alpha a_\alpha \xi_\alpha \left(\bar{\partial}\bar{\xi}_\alpha\right)^{1/2} \right), \qquad \alpha = 1, 2. \tag{3.7}$$

The first factor on the right hand side of (3.7) is just p. Hence the second factor of (3.7) is the expression given by (3.4) for the function κ_α. Thus (3.7) is the second equation in the KL system, $\bar{\partial}\tau_\alpha = -p\kappa_\alpha$, which completes the proof. ■

Now we discuss in detail certain classes of solutions to (1.1) that can be obtained from the transformation (1.5) by subjecting the systems (2.1) and (2.2) to the following algebraic constraints

$$|\xi_\alpha|^2 = 1, \qquad \alpha = 1, 2. \tag{3.8}$$

This implies that the functions ξ_1 and ξ_2 can differ only by a phase when these functions are represented in polar coordinates in the complex plane $\mathbb{C}$. By virtue of (3.8) equation (2.15) becomes

$$\partial\bar{\partial}\xi_\alpha + \partial\xi_\alpha \bar{\partial}\bar{\xi}_\alpha \xi_\alpha + 2\bar{\partial}\ln a_\alpha (\partial\xi_\alpha) = 0, \qquad \alpha = 1, 2. \tag{3.9}$$

Proposition 4. *Suppose that the functions ξ_α have unit modulus (3.8) and satisfy the overdetermined system composed of equations (2.1) and (3.9). Then the reciprocal functions ξ_α are solutions of equations (2.1) and (3.9).*

Proof. We wish to show that ξ_α is a solution of (3.9). The derivatives of ξ_α^{-1} are given by

$$\begin{aligned}&\partial(\xi_\alpha^{-1}) = -(\partial\xi_\alpha)\xi_\alpha^{-2}, \qquad \bar\partial(\bar\xi^{-1}) = -(\bar\partial\bar\xi_\alpha)\xi_\alpha^{-2},\\ &\bar\partial\partial\xi_\alpha^{-1} = -\bar\partial\partial\xi_\alpha\,\xi_\alpha^{-2} + 2(\partial\xi_\alpha)(\bar\partial\xi_\alpha)\xi_\alpha^{-3}.\end{aligned} \tag{3.10}$$

Substituting (3.10) into (2.1) we obtain

$$(-\partial\xi_1)\xi_1^{-2}(-\bar\partial\bar\xi_1)\bar\xi_1^{-2} = (-\partial\xi_2)\xi_2^{-2}(-\bar\partial\bar\xi_2)\bar\xi_2^{-2}$$

and, by straightforward computation, it is easy to show that, by using (3.8), the above equation is satisfied whenever ξ_α satisfies (2.1). Now we show that the reciprocal ξ_α^{-1} is a solution of (3.9). Substituting the derivatives (3.10) into (3.9) we obtain

$$\begin{aligned}&-(\bar\partial\partial\xi_\alpha)\xi_\alpha^{-2} + 2(\partial\xi_\alpha)(\bar\partial\xi_\alpha)\xi_\alpha^{-3} + (-\partial\xi_\alpha)\xi_\alpha^{-2}(-\bar\partial\bar\xi_\alpha)\bar\xi_\alpha^{-2}\xi_\alpha^{-1} + 2\bar\partial\ln a_\alpha(-\partial\xi_\alpha)\xi_\alpha^{-2}\\ &\quad = \xi_\alpha^{-2}[-(\bar\partial\partial\xi_\alpha) + 2(\partial\xi_\alpha)(\bar\partial\xi_\alpha)\xi_\alpha^{-1} + \xi_\alpha(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha) - 2\bar\partial\ln a_\alpha(\partial\xi_\alpha)].\end{aligned} \tag{3.11}$$

It is required to show that the expression (3.11) vanishes. We assume that the ξ_α satisfy (3.9). Then we can eliminate the second derivative $\bar\partial\partial\xi_\alpha$ by using the second order equation (3.9). Moreover from (3.8) we have that $\xi_\alpha = \bar\xi_\alpha^{-1}$. Differentiating both sides of this equation with respect to $\bar\partial$ we have

$$\bar\partial\xi_\alpha = -(\bar\partial\bar\xi_\alpha)\bar\xi_\alpha^{-2}.$$

Substituting the above equation for $\bar\partial\xi_\alpha$ and (3.9) into (3.11) we obtain

$$\begin{aligned}&(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha)\xi_\alpha + 2\bar\partial\ln a_\alpha(\partial\xi_\alpha) + 2(\partial\xi_\alpha)(\bar\partial\xi_\alpha)\xi_\alpha^{-1} + \xi_\alpha(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha) - 2\bar\partial\ln a_\alpha(\partial\xi_\alpha)\\ &\quad = -2(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha)\bar\xi_\alpha^{-1} + 2\xi_\alpha(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha) = 2\bar\xi_\alpha(-(\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha) + (\partial\xi_\alpha)(\bar\partial\bar\xi_\alpha)) = 0,\end{aligned}$$

which vanishes identically. This completes the proof. ■

We now investigate the case in which all the derivatives of the functions ξ_α are specified.

Proposition 5. *Let the functions ξ_α have unit modulus and their derivatives satisfy the following differential constraints*

$$\partial\xi_\alpha = F_\alpha(z)\xi_\alpha, \qquad \bar\partial\xi_\alpha = -\bar F_\alpha(\bar z)\xi_\alpha, \tag{3.12}$$

where the complex valued functions $F_\alpha(z)$ of class C^1 have equal modulus

$$|F_1(z)|^2 = |F_2(z)|^2. \tag{3.13}$$

Then the conditions (2.1) and (2.2) are satisfied identically and, for any real constants a_α, the complex functions ψ_α and φ_α given by (2.14) generate solutions of the KL system (1.1).

Proof. By substitution (3.12) and their conjugates into (2.1) it is seen, using (3.8), that equation (2.2) is reduced to (3.13). Differentiating (3.12) we have that

$$(i) \quad \bar{\partial}\partial\xi_\alpha = F_\alpha(z)\bar{\partial}\xi_\alpha, \qquad (ii) \quad \partial\bar{\partial}\xi_\alpha = -\bar{F}_\alpha(\bar{z})\partial\xi_\alpha. \tag{3.14}$$

Substituting the second derivative (3.14 i) and (3.12) into (2.16 i), we have

$$\bar{\partial}\ln a_\alpha = -\frac{F_\alpha(z)\bar{\partial}\xi_\alpha}{2F_\alpha(z)\xi_\alpha} - \frac{\bar{F}_\alpha(\bar{z})\bar{\xi}_\alpha}{2}\xi_\alpha = \frac{1}{2}\bar{F}_\alpha(\bar{z}) - \frac{1}{2}\bar{F}_\alpha(\bar{z}) = 0.$$

By substitution the second derivative (3.14ii) and (3.12) in (2.16i), it follows that

$$\bar{\partial}\ln a_\alpha = -\frac{\bar{F}_\alpha(z)\partial\xi_\alpha}{2\partial\xi_\alpha} - \frac{\bar{F}_\alpha(z)\bar{\xi}_\alpha}{2}\xi_\alpha = 0.$$

The compatibility condition (2.2) is satisfied identically since $\bar{\partial}a_\alpha = 0$. So a_α is any real constant. By virtue of Proposition 2 the complex functions ψ_α and φ_α which are defined in terms of ξ_α and a_α by (2.18) satisfy the KL system (1.1). ■

We discuss now the possibility of constructing more general classes of solutions of KL system (1.1) which are based on nonlinear superpositions of elementary solutions of equation (3.9).

Proposition 6. *Consider two functionally independent solutions, $\xi_{1\alpha}$ and $\xi_{2\alpha}$, of equations (3.9), which are labelled with an additional index and $\alpha = 1, 2$. Suppose that the complex functions $\xi_{1\alpha}$ and $\xi_{2\alpha}$ have unit modulus*

$$|\xi_{\beta\alpha}|^2 = 1$$

for $\alpha, \beta = 1, 2$. Suppose also that there exist real valued functions, $a_\alpha(z, \bar{z})$, such that the equation

$$\bar{\partial}\partial\xi_{\beta\alpha} + (\partial\xi_{\beta\alpha})(\bar{\partial}\bar{\xi}_{\beta\alpha})\xi_{\beta\alpha} + \bar{\partial}\ln a_\alpha(\partial\xi_{\beta\alpha}) = 0 \tag{3.15}$$

and its respective complex conjugate equation hold. Then the products of the functions

$$\eta_\alpha = \xi_{1\alpha}\xi_{2\alpha} \tag{3.16}$$

have to satisfy the equations

$$\partial\bar{\partial}\eta_\alpha + (\partial\eta_\alpha)(\bar{\partial}\bar{\eta}_\alpha) + \bar{\partial}\ln a_\alpha(\partial\eta_\alpha) = 0, \qquad \alpha = 1, 2.$$

and their respective complex conjugate equations.

Proof. We show that η_α given by (3.16) satisfies (3.15). In fact differentiating the functions η_α successively we obtain

$$\partial\eta_\alpha = (\partial\xi_{1\alpha})\xi_{2\alpha} + \xi_{1\alpha}(\partial\xi_{2\alpha}), \qquad \bar{\partial}\bar{\eta}_\alpha = (\bar{\partial}\bar{\xi}_{1\alpha})\bar{\xi}_{2\alpha} + \bar{\xi}_{1\alpha}(\bar{\partial}\bar{\xi}_{2\alpha}),$$
$$\bar{\partial}\partial\eta_\alpha = (\bar{\partial}\partial\xi_{1\alpha})\xi_{2\alpha} + (\partial\xi_{1\alpha})(\bar{\partial}\xi_{2\alpha}) + (\bar{\partial}\xi_{1\alpha})(\partial\xi_{2\alpha}) + \xi_{1\alpha}(\bar{\partial}\partial\xi_{2\alpha}).$$

Substituting the first and second derivatives of η_α into equation (2.15), we obtain

$$\begin{aligned}
&\xi_{2\alpha}\bar\partial\partial\xi_{1\alpha} + (\partial\xi_{1\alpha})(\bar\partial\xi_{2\alpha}) + (\bar\partial\xi_{1\alpha})(\partial\xi_{2\alpha}) + \xi_{1\alpha}\bar\partial\partial\xi_{2\alpha} + ((\partial\xi_{1\alpha})\xi_{2\alpha} + \xi_{1\alpha}(\partial\xi_{2\alpha}))\\
&\quad \times ((\bar\partial\bar\xi_{1\alpha})\bar\xi_{2\alpha} + \bar\xi_{1\alpha}(\bar\partial\bar\xi_{2\alpha}))\xi_{1\alpha}\xi_{2\alpha} + \bar\partial \ln a_\alpha(\partial\xi_{1\alpha}\xi_{2\alpha} + \xi_{1\alpha}\partial\xi_{2\alpha})\\
&\quad = \xi_{2\alpha}(\bar\partial\partial\xi_{1\alpha} + \xi_{1\alpha}(\partial\xi_{1\alpha})(\bar\partial\bar\xi_{1\alpha}) + (\partial\xi_{1\alpha})\bar\partial\ln a_\alpha) + \xi_{1\alpha}(\bar\partial\partial\xi_{2\alpha} + \xi_{2\alpha}(\partial\xi_{2\alpha})(\bar\partial\bar\xi_{2\alpha})\\
&\quad + \partial\xi_{2\alpha}\bar\partial\ln a_\alpha) + (\partial\xi_{1\alpha})(\bar\partial\xi_{2\alpha}) + (\bar\partial\xi_{1\alpha})(\partial\xi_{2\alpha}) + \xi_{1\alpha}^2(\partial\xi_{2\alpha})(\bar\partial\bar\xi_{1\alpha})\\
&\quad + \xi_{2\alpha}^2(\partial\xi_{1\alpha})(\bar\partial\bar\xi_{2\alpha}).
\end{aligned} \tag{3.17}$$

Since the first two terms appearing on the right hand side of (3.17) are individually equation (3.15) with $\beta = 1, 2$ respectively, these terms must vanish. So we show that the third term in (3.1) also vanishes. This follows by differentiating the relation (3.8). We obtain

$$\bar\partial\bar\xi_{\beta\alpha} = -\xi_{\beta\alpha}^{-2}(\bar\partial\xi_{\beta\alpha}).$$

The third term on the right hand side of (3.17) vanishes

$$(\partial\xi_{1\alpha})(\bar\partial\xi_{2\alpha}) + (\bar\partial\xi_{1\alpha})(\partial\xi_{2\alpha}) - (\partial\xi_{2\alpha})(\bar\partial\xi_{1\alpha}) - (\partial\xi_{1\alpha})(\bar\partial\xi_{2\alpha}) = 0.$$

This completes the proof. ■

The fact that the functions $\xi_{\beta\alpha}$ individually satisfy (2.1) for $\alpha, \beta = 1, 2$ is in itself not sufficient to guarantee that the product functions $\eta_\alpha = \xi_{1\alpha}\xi_{2\alpha}$ will also satisfy equation (2.1) due to the presence of the first derivatives of the functions η_α.

Proposition 7. *Suppose that the functions $\xi_{j,\alpha}$, $j = 1, 2$, satisfy condition (2.1) and have unit modulus (3.8). Then the product functions*

$$\eta_\alpha = \xi_{1\alpha}\xi_{2\alpha} \tag{3.18}$$

for $\alpha = 1, 2$, satisfy condition (2.1) provided that the following differential constraint holds for the functions $\xi_{\beta\alpha}$

$$\begin{aligned}
&(\partial\xi_{21})(\bar\partial\bar\xi_{11})\xi_{11}\bar\xi_{21} + \bar\xi_{11}\xi_{21}(\partial\xi_{11})(\bar\partial\bar\xi_{21})\\
&\quad = \xi_{12}\bar\xi_{22}(\partial\xi_{22})(\bar\partial\bar\xi_{12}) + \xi_{22}\bar\xi_{12}(\partial\xi_{12})(\bar\partial\bar\xi_{22}).
\end{aligned} \tag{3.19}$$

Proof. Suppose the set of functions ξ_{11}, ξ_{12} and ξ_{21}, ξ_{22}, satisfy (2.1). Then

$$(\partial\xi_{11})(\bar\partial\bar\xi_{11}) = (\partial\xi_{12})(\bar\partial\bar\xi_{12}), \qquad (\partial\xi_{21})(\bar\partial\xi_{21}) = (\partial\xi_{22})(\bar\partial\bar\xi_{22}).$$

Evaluating from (3.18) the expression $(\partial\eta_1)(\bar\partial\bar\eta_1) - (\partial\eta_2)(\bar\partial\bar\eta_2)$, which is equivalent to (2.1) for the functions η_α, and next using equations (3.8), we obtain

$$\begin{aligned}
&\partial(\xi_{11}\xi_{21})\bar\partial(\bar\xi_{11}\bar\xi_{21}) - \partial(\xi_{12}\xi_{22})\bar\partial(\bar\xi_{12}\bar\xi_{22}) = ((\partial\xi_{11})\xi_{21} + \xi_{11}(\partial\xi_{21}))\\
&\quad \times ((\bar\partial\bar\xi_{11})\bar\xi_{21} + \bar\xi_{11}(\bar\partial\bar\xi_{21})) - ((\partial\xi_{12})\xi_{22} + \xi_{12}(\partial\xi_{22}))((\bar\partial\bar\xi_{12})\bar\xi_{22} + (\bar\xi_{12}\bar\partial)\bar\xi_{22})\\
&\quad = (\partial\xi_{21})(\bar\partial\bar\xi_{11})\xi_{11}\bar\xi_{21} + \bar\xi_{11}\xi_{21}(\partial\xi_{11})(\bar\partial\bar\xi_{21}) - \xi_{12}\bar\xi_{22}(\partial\xi_{22})(\bar\partial\bar\xi_{12})\\
&\quad - \xi_{22}\bar\xi_{12}(\partial\xi_{12})(\bar\partial\bar\xi_{22}).
\end{aligned}$$

This vanishes whenever (3.19) is satisfied. ■

Propositions 6 and 7 can be used to give a criterion which enable us to determine whether a product of solutions of (3.9) is also a solution as well. We denote the pair of functions $\xi_{i\alpha}$ in the case in which they are equal for $\alpha = 1, 2$ as follows $\xi_{i1} = \xi_{i2} = \xi_i$.

Now we show that multisoliton solutions to the KL system (1.1) can be constructed based on nonlinear superposition of n elementary solutions of the system composed of (2.1) and (2.2).

Proposition 8. *(Factorization) Suppose that each function ξ_i for $i = 1, \ldots, n$ has unit modulus (3.8), and satisfies conditions (2.1) and (2.2). Suppose also that the functions ξ_i are solutions of the following differential constraint,*

$$\bar{\partial}\partial\xi_{i\alpha} + 2\frac{(\partial\xi_{i\alpha})(\bar{\partial}\bar{\xi}_{i\alpha})}{1+|\xi_{i\alpha}|^2}\xi_{i\alpha} + 2(\bar{\partial}\ln a)(\partial\xi_{i\alpha}) = 0 \tag{3.20}$$

and its complex conjugate equation, where a is a real-valued function of z and $\bar{z}$. Then the product functions η_α defined by

$$\eta_1 = \eta_2 = \prod_{k=1}^{n} \xi_k \tag{3.21}$$

satisfy the overdetermined system of equations (2.1), (2.2) and (3.20). The functions η_α determine a multisolitonic type solution to the KL system (1.1) by means of equations (2.18).

Proof. The proof is by induction. Consider two solutions ξ_1 and ξ_2, each of unit modulus, which both satisfy equation (3.20) under the same real valued function $a(z, \bar{z})$ for both ξ_i, $i = 1, 2$. Then, from Proposition 6, the functions η_α, which are given by the product $\eta_1 = \eta_2 = \xi_{1\alpha}\xi_{2\alpha}$, satisfy (3.20) under the same real valued function $a(z, \bar{z})$. Moreover, $\xi_{11} = \xi_{12}$ and $\xi_{21} = \xi_{22}$ holds. The condition (2.1) is identically satisfied. Equation (2.2) is the compatibility condition for the function a_α. Since the function $a(z, \bar{z})$ is common for the entire set of ξ_α, the compatibility condition (2.2) is also satisfied identically. Thus the product of $\xi_{1\alpha}$ and $\xi_{2\alpha}$ determines a solution to the KL system (1.1) by Proposition 2. By induction suppose that the functions

$$q_{n-1,1} = q_{n-1,2} = \prod_{k=1}^{n-1} \xi_{k,\alpha} = \prod_{k=1}^{n} \xi_k,$$

satisfy the hypotheses of the theorem up to some n. This means that $q_{n-1,\alpha} = q_{n-1}$ is a solution of (3.20) for the same real-valued function $a(z, \bar{z})$, and satisfies (2.1) and (2.2). The new product function $\xi_{n,\alpha} = \xi_n$ also satisfies the hypotheses of Proposition 8, as well as (2.1) and (2.2). Thus it follows by applying Proposition 6 that the new product function

$$\eta_\alpha = \left(\prod_{k=1}^{n-1} \xi_{k,\alpha}\right)\xi_{n,\alpha} = \prod_{k=1}^{n} \xi_{k,\alpha}$$

is also a solution of (3.19) under the same function $a(z, \bar{z})$. This function η_α clearly has unit modulus. If we set

$$\eta_1 = \eta_2 = \prod_{k=1}^{n} \xi_k,$$

then η_α satisfies (2.1), (2.2) as well as (3.10). Hence the functions η_α can be used to generate new multi-soliton type solutions of KL system (1.1) by a straightforward application of Proposition 2 to yield new solutions in the form,

$$\varphi_1 = \varphi_2 = a(z,\bar{z}) \left(\partial \prod_{k=1}^{n-1} \xi_k \right)^{1/2}, \quad \psi_1 = \psi_2 = a(z,\bar{z}) \prod_{k=1}^{n-1} \xi_k \left(\bar{\partial} \prod_{k=1}^{n-1} \bar{\xi}_k \right)^{1/2}. \tag{3.22}$$

■

4 Reduction of the Konopelchenko–Landolfi system to decoupled CP^1 sigma models

Now we discuss the case in which KL system (1.1) is subjected to a single differential constraint and its conjugate. This allows us to reduce this system to one which is composed of two decoupled CP^1 sigma model equations.

We start by introducing the new dependent variable

$$J = -\frac{1}{2}\left[\frac{\partial\xi_1 \partial\bar{\xi}_1}{(1+|\xi_1|^2)^2} + \frac{\partial\xi_2 \partial\bar{\xi}_2}{(1+|\xi_2|^2)^2} \right]. \tag{4.1}$$

Proposition 9. *The overdetermined system composed of equations (1.8) and the differential constraint $\bar{\partial}J = 0$ are consistent if and only if the complex functions ξ_1 and ξ_2 satisfy the decoupled pair of CP^1 sigma model equations*

$$\bar{\partial}\partial\xi_\alpha - \frac{2\bar{\xi}_\alpha}{1+|\xi_\alpha|^2}\bar{\partial}\xi_\alpha \partial\xi_\alpha = 0, \qquad \alpha = 1,2 \tag{4.2}$$

and their complex conjugates.

Proof. Differentiation of J with respect to $\bar{\partial}$ and imposing the differential constraint $\bar{\partial}J = 0$, we obtain the relation

$$\begin{aligned} \bar{\partial}J = -\frac{1}{2}\Bigg[& \frac{\bar{\partial}\partial\xi_1\, \partial\bar{\xi}_1}{(1+|\xi_1|^2)^2} + \frac{\partial\xi_1 \bar{\partial}\partial\bar{\xi}_1}{(1+|\xi_1|^2)^2} - 2\frac{\partial\xi_1 \partial\xi_1}{(1+|\xi_1|^2)^3}(\bar{\partial}\xi_1\bar{\xi}_1 + \xi_1\bar{\partial}\bar{\xi}_1) \\ & + \frac{\bar{\partial}\partial\xi_2\, \partial\bar{\xi}_2}{(1+|\xi_2|^2)^2} + \frac{\partial\xi_2\, \bar{\partial}\partial\bar{\xi}_2}{(1+|\xi_2|^2)^2} - 2\frac{\partial\xi_2 \partial\bar{\xi}_2}{(1+|\xi_2|^2)^3}(\bar{\partial}\xi_2\bar{\xi}_2 + \xi_2\bar{\partial}\bar{\xi}_2)\Bigg] = 0, \end{aligned} \tag{4.3}$$

and its complex conjugate. Solving the overdetermined system of equations (1.8) and (4.3) for the second derivatives $\partial\bar{\partial}\xi_1$, $\partial\bar{\partial}\xi_2$ and their conjugates, we find that the functions ξ_1 and ξ_2 satisfy equations (4.2) and their respective conjugates, respectively. ■

Note that all solutions of the $\mathbb{C}P^1$ model are well known [16]. They fall into three classes, those described by holomorphic or antiholomorphic functions and the mixed ones. The purpose for constructing solutions to the KL system (1.1) can be reduced to the following. Take any two functionally independent solutions of the sigma model (4.2) and substitute these solutions into equations (2.18). Then the functions a_α are determined by transformations (2.16). Hence, by virtue of Propositions 2 and 9, the functions ψ_α and φ_α thus obtained are solutions of the KL system (1.1).

5 Multisoliton solutions of the Konopelchenko–Landolfi system

At this point, we make use of the propositions presented in Section 3 and 4 in order to construct several new classes of solutions to the KL system, including multisoliton solutions.

Now we discuss several classes of solutions which can be obtained directly from equations (2.1), (2.2) and transformation (2.18).

1. We start with a simple class of analytic solutions to equations (2.1) and (2.2) of the exponential type

$$\xi_\alpha = e^z.$$

The successive derivatives of ξ_α are

$$\partial\xi_\alpha = e^z, \qquad \bar{\partial}\bar{\xi}_\alpha = e^{\bar{z}}, \qquad \bar{\partial}\partial\xi_\alpha = 0.$$

Condition (2.1) is identically satisfied and, from (2.16), the real functions a_α are determined by

$$\begin{aligned} \bar{\partial}\ln a_\alpha &= -\frac{e^{\bar{z}}}{1+e^{z+\bar{z}}}e^z = -\bar{\partial}\ln(1+e^{z+\bar{z}}), \\ \partial\ln a_\alpha &= -\frac{e^{z}}{1+e^{z+\bar{z}}}e^{\bar{z}} = -\partial\ln(1+e^{z+\bar{z}}). \end{aligned} \tag{5.1}$$

The compatibility condition for (5.1) is satisfied identically. Thus equations (5.1) are exact differentials and the functions a_α have the form

$$a_\alpha(z,\bar{z}) = c_\alpha(1+e^{z+\bar{z}})^{-1},$$

which are real-valued functions when c_α are real constants.

From Proposition 2 the functions φ_α and ψ_α are given by

$$\varphi_\alpha = c_\alpha(1+e^{z+\bar{z}})e^{z/2}, \qquad \psi_\alpha = c_\alpha(1+e^{z+\bar{z}})e^z e^{\bar{z}/2}. \tag{5.2}$$

The equation for a CMC-surface immersed into $\mathbb{R}^4$ can be obtained by substituting solutions (5.2) into (1.3). By eliminating the parameter $t = e^{z+\bar{z}}$ from the pair of equations

$$X_1^2 + X_2^2 = t(t-1)^2\left(3+3t+t^2\right)^2, \qquad X_3^2 + X_4^2 = 4(1+t)^6$$

an algebraic equation is found which describes the surface under consideration.

2. A more general class of exponential type solutions of (2.1) and (2.2) are provided by the analytic functions ξ_α of the form

$$\xi_\alpha = e^{i\phi_\alpha(z,\bar{z})}, \tag{5.3}$$

where the ϕ_α are real-valued functions. The derivatives of ξ_α are given by

$$\partial\xi_\alpha = i\partial\phi_\alpha e^{i\phi_\alpha}, \qquad \bar{\partial}\bar{\xi}_\alpha = -i\bar{\partial}\phi_\alpha e^{-i\phi_\alpha}.$$

By the use of (5.3), condition (2.1) becomes

$$(\partial\phi_1)(\bar\partial\phi_1) = (\partial\phi_2)(\bar\partial\phi_2) \tag{5.4}$$

which holds whenever $\partial\phi_1$ and $\partial\phi_2$ differ only by a phase. Substituting (5.3) into equation (2.16), we get

$$\bar\partial \ln a_\alpha = -\frac{\partial\bar\partial\phi_\alpha}{2\partial\phi_\alpha} - \frac{i}{2}\bar\partial\phi_\alpha + \frac{i}{2}\bar\partial\phi_\alpha = -\frac{\partial\bar\partial\phi_\alpha}{2\partial\phi_\alpha} \tag{5.5}$$

and its conjugate equation. Thus the compatibility condition for (5.5) requires that

$$\mathrm{Im}\left(-\frac{\partial\partial\bar\partial\phi_\alpha}{2\partial\phi_\alpha} + \frac{\partial\bar\partial\phi_\alpha}{2(\partial\phi_\alpha)^2}\,\partial^2\phi_\alpha\right) = 0 \tag{5.6}$$

holds. Thus, the functions ξ_α will provide a solution of (1.1) through (2.18) if condition (5.5) as well as (5.1) holds. In particular, if the function ϕ_α is a real-valued harmonic function,

$$\partial\bar\partial\phi_\alpha = 0, \tag{5.7}$$

the compatibility condition (5.6) is satisfied identically. Thus the corresponding solutions for the KL system (1.1) are given by the following expressions

$$\varphi_\alpha = a_\alpha(i\partial\phi_\alpha e^{i\phi_\alpha})^{1/2}, \qquad \psi_\alpha = a_\alpha e^{i\phi_\alpha}(-i\bar\partial\phi_\alpha e^{-i\phi_\alpha})^{1/2}. \tag{5.8}$$

For example, taking harmonic $\phi_1 = \phi_2 = (z^2 + \bar z^2)$, the corresponding surface is a cylinder with X_3 as its symmetry axis.

3. Consider a monomial class of solutions of (2.1) and (2.2) which are generated by ξ_α of the form

$$\xi_\alpha = \left[\frac{z - z_0}{\lambda_\alpha}\right]^n, \tag{5.9}$$

where λ_α and z_0 are arbitrary complex numbers. The derivatives of (5.9) are given by

$$\partial\xi_\alpha = \frac{n}{\lambda_\alpha^n}(z - z_0)^{n-1}, \qquad \bar\partial\bar\xi_\alpha = \frac{n}{\bar\lambda_\alpha^n}(\bar z - \bar z_0)^{n-1}, \qquad \bar\partial\partial\xi_\alpha = 0. \tag{5.10}$$

It is easy to compute constraint (2.1), which will be satisfied for all z provided that $|\lambda_\alpha|^2 = 1$ holds. Substituting derivatives (5.10) into (2.16), we obtain

$$\begin{aligned}
\bar\partial \ln a_\alpha &= -\frac{n\bar\lambda_\alpha^{-n}(\bar z - \bar z_0)^{n-1}}{1 + |z - z_0|^{2n}}\frac{(z - z_0)^n}{\lambda_\alpha^n} = -\bar\partial \ln\left(1 + |z - z_0|^{2n}\right),\\
\partial \ln a_\alpha &= -\frac{n\lambda_\alpha^{-n}(z - z_0)^{n-1}}{1 + |z - z_0|^{2n}}\frac{(\bar z - \bar z_0)^n}{\bar\lambda_\alpha^n} = -\partial \ln\left(1 + |z - z_0|^{2n}\right).
\end{aligned} \tag{5.11}$$

The compatibility condition for (5.11) is identically satisfied and, integrating (5.11), we obtain

$$a_\alpha(z, \bar z) = c_\alpha\left(1 + |z - z_0|^{2n}\right)^{-1}, \qquad c_\alpha \in \mathbb{R}. \tag{5.12}$$

Substituting (5.12) into (2.18), we obtain solutions φ_α and ψ_α of the KL system (1.1),

$$\varphi_\alpha = c_\alpha \left(1 + |z - z_0|^{2n}\right)^{-1} \left(\frac{n}{\lambda_\alpha^n}(z - z_0)^{n-1}\right)^{1/2},$$

$$\psi_\alpha = c_\alpha \left(1 + |z - z_0|^{2n}\right)^{-1} \left(\frac{z - z_0}{\lambda_\alpha}\right)^n \left(\frac{n}{\bar{\lambda}_\alpha^n}(\bar{z} - \bar{z}_0)^{n-1}\right)^{1/2}. \tag{5.13}$$

By eliminating the parameter $t = |z - z_0|^{2n}$ from the following pair of equations

$$X_1^2 + X_2^2 = (1 + t)^{-2} t \left(1 - t^{-1}\right)^2, \qquad X_3^2 + X_4^2 = 4(1 + t)^{-2}$$

we can determine the associated surface for solutions (5.13).

4. Now we discuss the construction of a class of solution of (2.1) and (2.2) which admits two arbitrary functions of one variable $f_\alpha(z)$,

$$\xi_\alpha = \frac{f_\alpha(z)}{\bar{f}_\alpha(\bar{z})}, \qquad \alpha = 1, 2. \tag{5.14}$$

It is clear from (5.11) that $|\xi_\alpha|^2 = 1$ holds, and the derivatives of (5.14) are given by

$$\partial \xi_\alpha = \frac{\partial f_\alpha}{\bar{f}_\alpha}, \qquad \bar{\partial} \partial \xi_\alpha = -\frac{\partial f_\alpha}{\bar{f}_\alpha^2} \bar{\partial} \bar{f}_\alpha$$

and their respective conjugates. Then, from equation (2.16), we find that

$$\bar{\partial} \ln a_\alpha = \frac{1}{2} \frac{\partial f_\alpha}{\bar{f}_\alpha^2} \bar{\partial} \bar{f}_\alpha \frac{\bar{f}_\alpha}{\partial f_\alpha} - \frac{1}{2} \frac{\bar{\partial} \bar{f}_\alpha}{f_\alpha} \frac{f_\alpha}{\bar{f}_\alpha} = \frac{1}{2} \left(\frac{\bar{\partial} \bar{f}_\alpha}{\bar{f}_\alpha} - \frac{\bar{\partial} \bar{f}_\alpha}{\bar{f}_\alpha}\right) = 0, \tag{5.15}$$

as well as its conjugate. So a_α is any real constant. Thus the compatibility condition for (5.15) is satisfied identically and equation (2.1) is satisfied provided that

$$\frac{|\partial f_1|^2}{|f_1|^2} = \frac{|\partial f_2|^2}{|f_2|^2}$$

holds. Thus, from (2.18), for any real constants a_α we have

$$\varphi_\alpha = a_\alpha \left(\frac{\partial f_\alpha}{\bar{f}_\alpha}\right)^{1/2}, \qquad \psi_\alpha = a_\alpha \frac{f_\alpha}{\bar{f}_\alpha} \left(\frac{\bar{\partial} \bar{f}_\alpha}{f_\alpha}\right)^{1/2}.$$

As an example, if we take $f(z) = z^n$, from (1.3) the corresponding surface can be calculated explicitly and is a cylinder with symmetry axis X_3.

5. Consider a class of rational solutions of the KL system (1.1) which admits simple poles based on transformation (1.5),

$$\xi_\alpha = \frac{z - b_\alpha}{\bar{z} - \bar{b}_\alpha}, \qquad \alpha = 1, 2, \qquad b_\alpha \in \mathbb{C}. \tag{5.16}$$

Equations (5.16) obey the constraint $|\xi_\alpha|^2 = 1$. The derivatives of ξ_α are given by

$$\partial \xi_\alpha = \frac{1}{\bar{z} - \bar{b}_\alpha}, \qquad \bar{\partial} \bar{\xi}_\alpha = \frac{1}{z - b_\alpha}, \qquad \bar{\partial} \partial \xi_\alpha = -\frac{1}{(\bar{z} - \bar{b}_\alpha)^2}. \tag{5.17}$$

Then condition (2.1) is reduced to the following

$$(z-b_1)(\bar{z}-\bar{b}_1)=(z-b_2)(\bar{z}-\bar{b}_2). \tag{5.18}$$

Equation (5.18) is satisfied when $b_1=b_2=b$. Equation (2.16) reduces to

$$\partial a_\alpha=0, \qquad \bar{\partial} a_\alpha=0 \tag{5.19}$$

and a_α is any real constant. Hence, applying Proposition 2, we obtain solutions of the KL system (1.1) of the form

$$\varphi_\alpha=a_\alpha\left(\frac{1}{\bar{z}-\bar{b}}\right)^{1/2}, \qquad \psi_\alpha=a_\alpha\left(\frac{z-b}{\bar{z}-\bar{b}}\right)\left(\frac{1}{z-b}\right)^{1/2}. \tag{5.20}$$

The surface in this case is a cylinder with X_3 the symmetry axis.

6. By the use of Proposition 2, an interesting class of periodic solutions to the KL system (1.1) satisfying the algebraic constraint (3.8) can be constructed. This class of solution can be determined by periodic functions ξ_α of the form

$$\xi_\alpha=\exp(\cos(z-b_\alpha)-\cos(\bar{z}-\bar{b}_\alpha)), \qquad b_\alpha\in\mathbb{C}, \qquad \alpha=1,2, \tag{5.21}$$

which satisfy (3.8). Their successive derivatives are given by

$$\begin{aligned}
&\partial\xi_\alpha=-\sin(z-b_\alpha)\exp(\cos(z-b_\alpha)-\cos(\bar{z}-\bar{b}_\alpha)),\\
&\bar{\partial}\bar{\xi}_\alpha=-\sin(\bar{z}-\bar{b}_\alpha)\exp(\cos(\bar{z}-\bar{b}_\alpha)-\cos(z-b_\alpha),\\
&\bar{\partial}\partial\xi_\alpha=\sin(z-b_\alpha)\sin(\bar{z}-\bar{b}_\alpha)\exp(\cos(z-b_\alpha)-\cos(\bar{z}-\bar{b}_\alpha)).
\end{aligned} \tag{5.22}$$

Equation (2.1) becomes

$$|\sin(z-b_1)|^2=|\sin(z-b_2)|^2. \tag{5.23}$$

Equation (5.23) holds when $b_1=b_2=b$. Substituting the derivatives (5.22) into (2.16), we find that the functions a_α satisfy the following conditions

$$\partial\ln a_\alpha=\sin(\bar{z}-\bar{b}_\alpha), \qquad \bar{\partial}\ln a_\alpha=\sin(z-b_\alpha). \tag{5.24}$$

Integrating (5.24), we get

$$a_\alpha=\exp(c_\alpha(\sin(z-b_\alpha)+\sin(\bar{z}-\bar{b}_\alpha))). \tag{5.25}$$

If $b_1=b_2=b\in\mathbb{C}$, equations (2.18) lead to the following nontrivial, periodic solutions of the KL system (1.1),

$$\begin{aligned}
\varphi_\alpha&=\exp(c_\alpha(\sin(z-b)+\sin(\bar{z}-\bar{b})))(-\sin(z-b)\exp(\cos(z-b)-\cos(\bar{z}-\bar{b})))^{1/2},\\
\psi_\alpha&=\exp(c_\alpha(\sin(z-b)+\sin(\bar{z}-\bar{b})))\exp(\cos(z-b)-\cos(\bar{z}-\bar{b}))\\
&\quad\times(-\sin(\bar{z}-\bar{b})\exp(\cos(\bar{z}-\bar{b})-\cos(z-b)))^{1/2}.
\end{aligned} \tag{5.26}$$

The corresponding surface is a cylinder with X_3 as symmetry axis.

7. Another class of solutions of the KL system can be obtained by replacing the cosine function in (5.21) by the hyperbolic function cosh such that functions ξ_α satisfy (3.8)

$$\xi_\alpha = \exp(\cosh(z - b_\alpha) - \cosh(\bar{z} - \bar{b}_\alpha)). \tag{5.27}$$

The successive derivatives of ξ_α are given by

$$\begin{aligned}
\partial\xi_\alpha &= \sinh(z - b_\alpha)\exp(\cosh(z - b_\alpha) - \cosh(\bar{z} - \bar{b}_\alpha)),\\
\bar{\partial}\bar{\xi}_\alpha &= \sinh(\bar{z} - \bar{b}_\alpha)\exp(\cosh(\bar{z} - \bar{b}_\alpha) - \cosh(z - b_\alpha)),\\
\bar{\partial}\partial\xi_\alpha &= -\sinh(\bar{z} - \bar{b}_\alpha)\sinh(z - b_\alpha)\exp(\cosh(z - b_\alpha) - \cosh(\bar{z} - \bar{b}_\alpha)).
\end{aligned} \tag{5.28}$$

Condition (2.1) then takes the form

$$\begin{aligned}
&|\sinh(z - b_1)\exp(\cosh(z - b_1) - \cosh(\bar{z} - \bar{b}_1))|\\
&\qquad = |\sinh(z - b_2)\exp(\cosh(z - b_2) - \cosh(\bar{z} - \bar{b}_2))|.
\end{aligned} \tag{5.29}$$

Equation (5.29) is satisfied when $b_1 = b_2 = b \in \mathbb{C}$. Equations (2.16) become identical to (5.19) in this case. Hence a_α are any real constants. Therefore, by the use of (2.18) the required solutions of KL system (1.1) are given by

$$\begin{aligned}
\varphi_{1,2} &= a_\alpha(\sinh(z - b)\exp(\cosh(z - b) - \cosh(\bar{z} - \bar{b})))^{1/2},\\
\psi_{1,2} &= a_\alpha \exp(\cosh(z - b) - \cosh(\bar{z} - \bar{b}))\\
&\qquad \times (\sinh(\bar{z} - \bar{b})\exp(\cosh(\bar{z} - \bar{b}) - \cosh(z - b)))^{1/2}.
\end{aligned} \tag{5.30}$$

These solutions represent a bump type solution and the corresponding surface is a cylinder with symmetry axis X_3. Note that solutions which yield cylinders, such as (5.8), (5.20), (5.26) and (5.30), have applications to certain types of cosmological models and are useful for describing event horizons in general relativity [4].

8. Another class of hyperbolic solutions is obtained by using the tanh function instead of cosh in expression (5.21) and is generated by

$$\xi_\alpha = \exp(\tanh(z - b_\alpha) - \tanh(\bar{z} - b_{\bar{\alpha}})).$$

The successive derivatives of ξ_α are given by

$$\begin{aligned}
\partial\xi_\alpha &= \operatorname{sech}^2(z - b_\alpha)\exp(\tanh(z - b_\alpha) - \tanh(\bar{z} - \bar{b}_\alpha)),\\
\bar{\partial}\bar{\xi}_\alpha &= \operatorname{sech}^2(\bar{z} - \bar{b}_\alpha)\exp(\tanh(\bar{z} - \bar{b}_\alpha) - \tanh(z - b_\alpha)),\\
\bar{\partial}\partial\xi_\alpha &= -\operatorname{sech}^2(z - b_\alpha)\operatorname{sech}^2(\bar{z} - \bar{b}_\alpha)\exp(\tanh(z - b_\alpha) - \tanh(\bar{z} - \bar{b}_\alpha)).
\end{aligned}$$

Condition (2.1) takes the form

$$\begin{aligned}
&|\operatorname{sech}^2(z - b_1)\exp(\tanh(z - b_1) - \tanh(\bar{z} - \bar{b}_1))|\\
&\qquad = |\operatorname{sech}^2(z - b_2)\exp(\tanh(z - b_2) - \tanh(\bar{z} - \bar{b}_2))|
\end{aligned} \tag{5.31}$$

which holds whenever $b_1 = b_2 = b$. Equations (2.16) are reduced to (5.19), which implies that the a_α are real constants. Thus from (2.18) we can write

$$\begin{aligned}
\varphi_\alpha &= a(\operatorname{sech}^2(z - b)\exp(\tanh(z - b) - \tanh(\bar{z} - \bar{b})))^{1/2},\\
\psi_\alpha &= a\exp(\tanh(z - b) - \tanh(\bar{z} - \bar{b})))\\
&\qquad \times (\operatorname{sech}^2(\bar{z} - \bar{b})\exp(\tanh(\bar{z} - \bar{b}) - \tanh(z - b)))^{1/2}, \qquad \alpha = 1, 2.
\end{aligned}$$

This type of solution represents a kink-type solution. Substituting these in (1.3), the corresponding surface is given by a cylinder with X_3 as its symmetry axis.

9. Finally we discuss a class of solutions for the KL system (1.1) admitting simple poles which can be constructed by applying Proposition 8, namely,

$$\xi_{k1} = \xi_{k2} = \frac{z - b_k}{\bar{z} - \bar{b}_k}, \qquad b_k \in \mathbb{C}. \tag{5.32}$$

These functions $\xi_{k\alpha}$ have unit modulus and identically satisfy condition (2.1) and equation (3.20) for any real constants a_α. By virtue of Proposition 8, a solution of KL system (1.1) is generated by the functions η_α in the form of the product,

$$\eta_1 = \eta_2 = \prod_{k=1}^{n} \frac{z - b_k}{\bar{z} - \bar{b}_k}. \tag{5.33}$$

Substituting (5.33) into (2.18), we obtain

$$\varphi_\alpha = a \left(\eta_\alpha \sum_{k=1}^{n} \frac{1}{z - b_k} \right)^{1/2}, \qquad \psi_\alpha = a\eta_\alpha \left(\bar{\eta}_\alpha \sum_{k=1}^{n} \frac{1}{\bar{z} - \bar{a}_k} \right)^{1/2}.$$

Note that this type of solution admits only simple poles and represents multisoliton solutions of the KL system (1.1). The associated surface for $n = 1$ is a cylinder.

With regard to the examples presented here, we mention two configurations which are different from instantons and have applications in string theory [4].

The developable surfaces are the simplest. These surfaces have Gaussian curvature $K = 0$, satisfy (1.1) and the additional constraint

$$\left(|\psi_1|^2 + |\varphi_1|^2\right)\left(|\psi_2|^2 + |\varphi_2|^2\right) = |A(z)|^2, \tag{5.34}$$

where $A(z)$ is an arbitrary holomorphic function.

A second class of surfaces, those with flat normal bundle, correspond to vanishing normal curvature $K_N = 0$. These surfaces are generated by the system (1.1) subject to the additional constraint

$$\left(|\psi_2|^2 + |\varphi_2|^2\right) = |A(z)|^2 \left(|\psi_1|^2 + |\varphi_1|^2\right), \tag{5.35}$$

where $A(z)$ is an arbitrary holomorphic function. If $A(z) = 1$ and we differentiate both sides of (5.35) with respect to ∂ and use (1.1) to eliminate the known derivatives of ψ_i and φ_i, we obtain the constraint

$$\bar{\partial}\psi_1\bar{\psi}_1 + \varphi_1\bar{\partial}\bar{\varphi}_1 = \bar{\partial}\psi_2\bar{\psi}_2 + \varphi_2\bar{\partial}\varphi_2.$$

Several solutions which have been presented in this Section fall into this second classification if the functions a_α are chosen properly. For example, the solutions (5.20) and (5.30) in examples 5 and 7 satisfy (5.35) for $A(z) = 1$ provided that $a_1 = a_2 = 1$. Another class of solution (5.26) obeys condition (5.35) when $c_1 = c_2 = 1$.

6 A simple model for strings in $\mathbb{R}^4$

According to [4], the unification of gravity and quantum mechanics seems to require a new formulation of physics at small distance scales. In the string approach elementary particles can be thought of as strings, and differ from all familiar quantum mechanical field theories the constituent particles of which are pointlike, whereas a string has extension in space-time [21, 22]. Moreover, superstring theory combines string theory with another mathematical structure namely supersymmetry. This theory makes it possible to consider all four fundamental forces as various aspects of a single underlying principle [23]. The forces are unified in a way determined by the requirement that the theory be internally consistent. The strings which are postulated by the theory would then be about 10^{20} times smaller than the diameter of the proton.

We introduce an independent world sheet metric, $g_{ab}(x, y)$, which is independent of the string variables. The Polyakov form of the Lagrangian density [24] is written as

$$L = -\frac{1}{4\pi\alpha}\sqrt{-g}g^{ab}\partial_a X_\mu \partial_b X^\mu, \qquad a, b = 1, 2 \qquad \mu = 1, \ldots, 4, \tag{6.1}$$

with action

$$S = \int dx\, dy\, L,$$

where $g = \det(g_{ab})$, and α is the square of the characteristic length scale of perturbative string theory. The tension of the fundamental string is $1/2\pi\alpha$. This is a generalization of the second order point-particle action. Notice that the Polyakov action resembles an action with scalar fields interacting with an external two-dimensional gravitational field [24]. It is worth noting here that the Polyakov form of the action is in fact equivalent to another form of the action which appears in string theory [24], namely, the Nambu–Goto action. To obtain this equivalence, the equation of motion which is derived by varying the metric can be used. This implies that

$$h_{ab}(-h)^{-1/2} = g_{ab}(-g)^{-1/2}, \tag{6.2}$$

where h_{ab} is the induced metric, so that g_{ab} is proportional to the induced metric. Equation (6.2) can be used to eliminate g_{ab} from the Polyakov action, thus giving the Nambu–Goto form. The independent world sheet metric g_{ab} is taken to be

$$g_{ab} = g^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Euler Lagrange equations of motion are obtained from (6.1) and have the form,

$$\partial_x \partial_y X_\mu = 0, \qquad \mu = 1, \ldots, 4. \tag{6.3}$$

Applying the identity $\partial_x \partial_y = i(\partial^2 - \bar{\partial}^2)$, equation (6.2) becomes

$$(\partial^2 - \bar{\partial}^2)X_\mu = 0. \tag{6.4}$$

Substituting the expressions for the position vector $\mathbf{X}$ given by (1.3) into (6.4), we obtain the following set of equations

$$\begin{aligned}
&\partial(\bar{\psi}_1\bar{\psi}_2+\varphi_1\varphi_2)+\bar{\partial}(\psi_1\psi_2+\bar{\varphi}_1\bar{\varphi}_2)=0,\\
&\partial(\bar{\psi}_1\bar{\psi}_2-\varphi_1\varphi_2)-\bar{\partial}(\psi_1\psi_2-\bar{\varphi}_1\bar{\varphi}_2)=0,\\
&\partial(\bar{\psi}_1\varphi_2+\bar{\psi}_2\varphi_1)-\bar{\partial}(\psi_1\bar{\varphi}_2+\psi_2\bar{\varphi}_1)=0,\\
&\partial(\bar{\psi}_1\varphi_2-\bar{\psi}_2\varphi_1)+\bar{\partial}(\psi_1\bar{\varphi}_2-\psi_2\bar{\varphi}_1)=0.
\end{aligned} \tag{6.5}$$

Adding the first two and last two equations in (6.5), respectively, we obtain

$$\partial(\bar{\psi}_1\bar{\psi}_2)+\bar{\partial}(\bar{\varphi}_1\bar{\varphi}_2)=0, \qquad \partial(\bar{\psi}_1\varphi_2)-\bar{\partial}(\psi_2\bar{\varphi}_1)=0. \tag{6.6}$$

Equations (6.6), together with their complex conjugates, are equivalent to (6.5). For example, taking the sum and difference of the first equation in (6.6) with its own conjugate, we obtain the first two equations in (6.5).

Now in this context we discuss a condition under which the KL system (1.1) becomes a linear system of equations.

Proposition 10. *Consider the overdetermined system composed of the KL-system (1.1) which is subjected to DCs of the form*

$$\psi_1\partial\bar{\psi}_1-\bar{\varphi}_1\partial\varphi_1+\psi_2\partial\bar{\psi}_2-\bar{\varphi}_2\partial\varphi_2=0. \tag{6.7}$$

Assume that all first order derivatives of ψ_α and φ_α with respect to z and $\bar{z}$ are expressible in terms of polynomial functions which depend on ψ_α and φ_α with constant coefficients. Then the KL system (1.1) subjected to (6.7) is equivalent to the following linear system of equations

$$\partial\begin{pmatrix}\psi_\alpha\\ \varphi_\alpha\end{pmatrix}=p_0\begin{pmatrix}\varphi_\alpha\\ -\epsilon\psi_\alpha\end{pmatrix}, \qquad \bar{\partial}\begin{pmatrix}\psi_\alpha\\ \varphi_\alpha\end{pmatrix}=p_0\begin{pmatrix}\epsilon\varphi_\alpha\\ -\psi_\alpha\end{pmatrix}, \qquad \epsilon=\pm1, \tag{6.8}$$

$$(u_1u_2)^{1/2}=p_0, \qquad |\psi_\alpha|^2+|\varphi_\alpha|^2=u_\alpha, \tag{6.9}$$

where the quantities p_0 and u_α are arbitrary real constants.

Proof. The aim is to find the explicit form of all first derivatives of ψ_α and φ_α in such a way that they do not provide any additional differential constraints on ψ_α and φ_α other than those in (6.7) when the compatibility conditions are added. In this case we can close the KL system (1.1) and show that the compatibility conditions for (1.1) and (6.7) under the assumption of Proposition 9 coincide only with the requirements (6.8) and (6.9). Indeed, using u_α defined in (1.1) the first and second derivatives of u_α, ψ_α and φ_α are

$$\begin{aligned}
&\partial u_\alpha=\psi_\alpha\partial\bar{\psi}_\alpha+\bar{\varphi}_\alpha\partial\varphi_\alpha,\\
&\partial\bar{\partial}u_\alpha=\partial\bar{\psi}_\alpha\bar{\partial}\psi_\alpha+\partial\varphi_\alpha\bar{\partial}\bar{\varphi}_\alpha-p^2u_\alpha, \qquad \alpha=1,2,
\end{aligned} \tag{6.10}$$

and

$$\begin{aligned}
&\bar{\partial}\partial\psi_\alpha=\bar{\varphi}_\alpha\partial p-p^2\bar{\psi}_\alpha, \qquad \partial^2\psi_\alpha=\varphi_\alpha\partial p+p\partial\varphi_\alpha,\\
&\bar{\partial}\partial\varphi_\alpha=-\psi_\alpha\partial p-p^2\varphi_\alpha, \qquad \bar{\partial}^2\varphi_\alpha=-\psi_\alpha\bar{\partial}p-p\bar{\partial}\psi_\alpha
\end{aligned} \tag{6.11}$$

and their respective complex conjugate equations.

Suppose that the unknown derivatives $(\bar{\partial}\psi_\alpha, \partial\bar{\psi}_\alpha, \partial\varphi_\alpha, \bar{\partial}\bar{\varphi}_\alpha)$ other than those appearing in (1.1) are polynomial in ψ_α and φ_α with constant coefficients. The analysis of the dominant terms in the variables ψ_α and φ_α for the compatibility conditions of all first order derivatives leads to the requirement that all unknown derivatives have to be cubic in terms of the fields ψ_α and φ_α and take the following specific form

$$\bar{\partial}\psi_\alpha = p\sum_{\alpha=1}^{2}(c_\alpha^1\psi_\alpha + c_\alpha^2\varphi_\alpha), \qquad \partial\varphi_\alpha = p\sum_{\alpha=1}^{2}(c_\alpha^3\psi_\alpha + c_\alpha^4\varphi_\alpha). \tag{6.12}$$

Here, c_α^i, $i = 1, \ldots, 4$, are complex constants to be determined from the compatibility conditions for (6.11) and (6.12). This leads us to a system of equations which are linear in ψ_α and φ_α and has a unique solution of the form (6.7). Differentiation of p with respect to z and $\bar{z}$ yields

$$\partial p = \frac{1}{2}p^{-1}(u_2\partial u_1 + u_1\partial u_2), \qquad \bar{\partial}p = \frac{1}{2}p^{-1}(u_2\bar{\partial}u_1 + u_1\bar{\partial}u_2). \tag{6.13}$$

Taking into account (6.8), we obtain that equations (6.13) vanish identically and conditions (6.9) hold. Note that under the condition (6.7), we show that the KL system (1.1) admits a conserved quantity with p a real constant. It implies by virtue of (1.4) that the Gaussian curvature $K = 0$ and so the surface is flat. ■

Note that on substitution of the derivatives of functions ψ_α and φ_α, given in (6.8) into the string equations (6.6), it is seen that (6.8) is a solution to the string equations. If time is regarded as a spatial dimension, as appropriate in $\mathbb{R}^4$, the world sheet of a closed string can be thought of as a surface that joins the string at its initial point and at the end of its spacetime path. It has been shown here that the Polyakov form of the action [4, 25] can be used in the KL formulation of surfaces in $\mathbb{R}^4$, where the motion of the string is such that the action is minimized. This variational principle yields the string equations of motion (6.6).

In conclusion it is worth noting that the study of stable classical solutions presented here is one of the most important problems in investigating quantum theories of strings, in particular, in constructing perturbation series about a known classical solution, $X_\mu \to X_{\mu,cl} + X_{\mu,q}$.

From the physical point of view, to perform calculations in the path integral approach, one often shifts the integration variable by a solution to the classical system. In order to extract physical predictions from a theory we must firstly quantize the theory and obtain the physical states. On account of symmetries present in such theories, there can be great redundancies in the degrees of freedom and so the quantization of the string can be nontrivial. A quantum theory which describes interactions of strings can be represented by a Feynman path integral. In this form of quantization the amplitudes are obtained by summing over all possible histories which interpolate between the initial and final states. Each path is weighted by the factor $\exp(iS_{cl})$, where S_{cl} is the classical action for the given history. An amplitude in string theory is defined by summing over all world-sheets connecting the initial and final curves, or surfaces. In the sum over world sheets the integral runs over all Euclidean metrics and over all embeddings of the surface determined by the position vector X^μ of the world sheet in spacetime. It is usual to take the action in

Euclidean form, which is more accurately defined as far as convergence of the path integral is concerned.

An analysis of solutions carried out in Section 4 can provide us with admissible surfaces with respect to the KL system (1.1) in $\mathbb{R}^4$, which can be used in the calculation of quantum corrections to classical results. Recently the GW representations for inducing minimal surfaces in pseudo-Riemannian multidimensional spaces has been formulated in [1, 15]. It should be possible to extend the path integral approach to these multidimensional spaces as well. This task will be undertaken in future work.

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References

[1] Konopelchenko B G and Landolfi G, Generalized Weierstrass Representation for Surfaces in Multi-Dimensional Riemann Spaces, *J. Geom. Phys.* **29** (1999), 319–333.

[2] Weierstrass K, Fortsetzung der Untersuchung über die Minimalflächen, Mathematische Werke, Vol. 3, Verlagsbuchhandlung, Hillesheim, 1866, 219–248.

[3] Enneper A, *Nachr. Königl. Gesell. Wissensch. Georg-Augustus-Univ., Göttingen* **12** (1868), 258.

[4] Gross D G, Pope C N and Weinberg S, Two-Dimensional Quantum Gravity and Random Surfaces, World Scientific, Singapore, 1992.

[5] Nelson D, Piran T and Weinberg S, Statistical Mechanics of Membranes and Surfaces, World Scientific, Singapore, 1992.

[6] Ou-Yang Zhong-Can, Liu Ji-Xing and Xie Yu-Zhang, Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases, World Scientific, Singapore, 1999.

[7] Kenmotsu K, Weierstrass Formula for Surfaces of Prescribed Mean Curvature, *Math. Ann.* **245** (1979), 89-99.

[8] Hoffman D A and Osserman R, The Gauss Map of Surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$, *Proc. London Math. Soc.* **50** (1985), 27–56.

[9] Osserman R, A Survey of Minimal Surfaces, Dover, New York, 1996.

[10] Budnich P and Rigoli M, Cartan Spinors, Minimal Surfaces and Strings, Il, *Nuovo Cimento* **102** (1988), 609–646.

[11] Konopelchenko B G and Taimanov I A, Constant Mean Curvature Surfaces via an Integrable Dynamical System, *J. Phys.* **A29** (1996), 1261–1265.

[12] Carroll R and Konopelchenko B G, Generalized Weierstrass–Enneper Inducing Conformal Immersions and Gravity, *Int. J. Mod. Phys.* **A11**, Nr. 7 (1996), 1183–1216.

[13] Bobenko I A and Eitner U, Painlevé Equations in the Differential Geometry of Surfaces, Lecture Notes in Mathematics, Vol. 1753, Berlin, 2000.

[14] Bobenko A I, Surfaces in Terms of 2 by 2 Matrices. Old and New Integrable Cases, in Aspects of Mathematics, Editors: A P Fordy and J C Wood, Vieweg, Wiesbaden, 1994.

[15] Konopelchenko B G and Landolfi G, Induced Surfaces and Their Integrable Dynamics II. Generalized Weierstrass Representations in 4-D Spaces and Deformations via DS Hierarchy, *Studies in Appl. Math.* **104** (1999), 129–168.

[16] Zakrzewski W, Low Dimensional Sigma Models, Hilger, Bristol, 1989.

[17] Martina L, Myrzakul K, Myrzakulov R and Soliani G, Deformation of Surfaces, Integrable Systems, and Chern–Simons Theory, *J. Math. Phys.* **42** (2001), 1397–1417.

[18] Martina L, Pashaev O K and Soliani G, Chern–Simons Gauge Field Theory of Two-Dimensional Ferromagnets, *Phys. Rev.* **B21** (1993), 15787–15791.

[19] Bracken P and Grundland A M, On Certain Classes of Solutions of the Weierstrass–Enneper System Inducing Constant Mean Curvature Surfaces, *J. Nonlin. Math. Phys.* **6** (1999), 294–313.

[20] Bracken P and Grundland A M, Symmetry Properties and Explicit Solutions of the Generalized Weierstrass System, *J. Math. Phys.* **42**, Nr. 3 (2001), 1250–1282.

[21] Konopelchenko B G, Soliton Curvatures of Surfaces and Spaces, *J. Math. Phys.* **38** (1997), 434–457.

[22] Konopelchenko B G and Landolfi G, On Classical String Configurations, *Mod. Phys. Lett.* **A12** (1997), 3161–3168.

[23] Konopelchenko B G and Landolfi G, On Rigid String Instantons in Four Dimensions, *Phys. Lett.* **B459** (1999), 522–526.

[24] Polchinski J, String Theory, Cambridge University Press, Cambridge, 1998.

[25] Taylor W, Matrix Theory: Matrix Quantum Mechanics as a Fundamental Theory, *Rev. Mod. Phys.* **73** (2001), 419–461.